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Wedge states in string field theory

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Abstract: The wedge states form an important subalgebra in the string field theory. We review and further investigate their various properties. We find in particular a novel expression for the wedge states, which allows to understand their star products purely algebraically. The method allows also for treating the matter and ghost sectors separately. It turns out, that wedge states with different matter and ghost parts violate the associativity of the algebra. We introduce and study also wedge states with insertions of local operators and show how they are useful for obtaining exact results about convergence of level truncation calculations. These results help to clarify the issue of anomalies related to the identity and some exterior derivations in the string field algebra.

Keywords: Bosonic Strings, Tachyon Condensation, String Field Theory.

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1. Introduction

Open string field theory [1]–[5] in the last two years has experienced great renaissance as it turned out to be a powerful tool for understanding nonperturbative off-shell phenomenon of tachyon condensation in string theory. The famous Sen's conjectures [6, 7], by now has been confirmed within the string field theory to a high level of confidence following the works [8, 9]. For a complete lists of references the reader is referred to the reviews [10]–[14]. Unfortunately, most of the checks in the Witten's cubic string field theory up to date were performed only numerically. Notable exception is the idea of the vacuum string field theory (VSFT) [15, 16], which is the standard string field theory expanded around the true nonperturbative vacuum. This approach seems quite promising, it has already been possible to obtain some analytic results as for example the ratios of D-brane tensions [17].

The basic assumption of VSFT is that after suitable reparametrization of the string field, the kinetic operator can be expressed entirely using the ghost oscillators. The classical solutions corresponding to D-branes can then be found in terms of projectors in the matter sector of the string field algebra. The only explicitly known (nontrivial) projector until very recently, has been the sliver state, which belongs to the family of wedge states, found by Rastelli and Zwiebach [18]. These states form a commutative subalgebra within the string field algebra. Among other states in this family, there is also the $\mathrm{SL}(2,\mathbb{R})$ invariant vacuum $|0\rangle$ and the identity element $|I\rangle$ of the algebra.

In this paper we would like to study various aspects of this important family of wedge states, expressing our results mainly in the so called universal basis formed by matter Virasoro operators and ghosts acting on the vacuum. We shall find a new expression for all the wedge states which enables us to prove algebraically the star multiplication rules for the wedge states. We can also express all the wedge states as explicit reparametrizations of the vacuum, which completes the geometrical picture of [16]. The sliver and the identity emerge as infinite reparametrizations of the vacuum and this property makes them projectors. The methods developed allow us to study so called unbalanced wedge states, which are factorized states whose matter and ghost parts correspond to different wedge states. We find that such states necessarily violate the cyclicity of the three vertex and therefore also the associativity. The same happens for wedge states in the twisted conformal field theory (CFT) of [16] whose overall central charge is nonvanishing.

Another topic we discuss thoroughly is star multiplication of wedge states with insertions of local operators. The results can be used to show, that in certain cases the level truncation actually breaks down. Second application of the rules for wedge states with insertions is to study the anomalies related to the string field algebra identity and some exterior derivations. We show in particular, that the object $c_0|I\rangle$ should be better excluded from the star algebra, since it has ambiguous star products with the wedge states. As a byproduct we find a new "sum rule" for the tachyon condensate in ordinary cubic string field theory.

The paper is organized as follows. In section 2 we review some basic facts about finite conformal transformation and wedge states. In section 3 we introduce wedge states with insertions and derive their star-multiplication properties. The results are used to demonstrate that level expansion breaks down in the product of just four Fock space states and that this could possibly violate associativity. Section 4 deals with the behavior of the coefficients which appear in the original definition [18] of the wedge states. It introduces the mathematical concept of the iterative logarithm, which helps in section 5 to find a novel explicit formula for all the wedge states. This can be also rewritten in a form which makes manifest that the wedge states are reparametrizations of the star product. Particular attention is also paid to matter and ghost sectors separately. We show that the wedge states in these sectors do not have finite norm and generically lead to violation of associativity.

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Finally section 6 is devoted to the star algebra identity. We show explicitly that the object $c_0|I\rangle$ has ambiguous star products with other states in the algebra. Appendix A illustrates how the multiplication of wedge states behaves in the level truncation approximation and appendix B contains some graphs related to the discussion in section 4.

2. Wedge states

2.1 Finite conformal transformation

Let us recall first some basic facts about finite conformal transformations. A primary field $\Psi(z)$ of conformal weight d transforms under finite conformal transformation f as

$$f \circ \Psi = \left[f'(z)\right]^d \Psi(f(z)) . \tag{2.1}$$

We would like to rewrite this transformation rule using the Virasoro generators L_n of the conformal group in the form

$$\left[f'(z)\right]^d \Psi(f(z)) = U_f \Psi(z) U_f^{-1} , \tag{2.2}$$

where

$$U_f = e^{\sum v_n L_n} . \tag{2.3}$$

To determine the coefficients v_n we note that

$$U_f \Psi(z) U_f^{-1} = e^{\mathrm{ad}_{\sum v_n L_n}} \Psi(z) , \tag{2.4}$$

where as usually $\mathrm{ad}_X Y = [X, Y]$. We may prove an important identity

$$\left(\mathrm{ad}_{\sum v_n L_n}\right)^k \Psi(z) = (v(z)\partial_z + dv'(z))^k \Psi(z) , \tag{2.5}$$

for any $k \in \mathbb{N}$, where we set

$$v(z) = \sum v_n z^{n+1} . \tag{2.6}$$

The proof for $k = 1$ can be easily performed for example by expanding $\Psi = \sum \Psi_n/(z^{n+d})$ and using the commutation relations

$$[L_m, \Psi_n] = ((d-1)m - n)\Psi_{m+n} . \tag{2.7}$$

For $k > 1$ it then extends trivially. We thus see that in general

$$U_f \Psi(z) U_f^{-1} = e^{v(z)\partial_z + dv'(z)} \Psi(z) . \tag{2.8}$$

Our task is now for a given $f(z)$ to find a solution $v(z)$, such that for any $\Psi(z)$ of an arbitrary dimension d holds

$$e^{v(z)\partial_z + dv'(z)} \Psi(z) = \left[f'(z)\right]^d \Psi(f(z)) . \tag{2.9}$$

A priori, it is not even clear that such $v(z)$ exists. Let us insert into the left hand side the identity $e^{-v\partial} e^{v\partial}$. Since

$$e^{v(z)\partial_z} \Psi(z) = \Psi(e^{v(z)\partial_z} z) , \tag{2.10}$$

as one can easily check, and $e^{v(z)\partial_z + dv'(z)} e^{-v(z)\partial_z}$ is just an ordinary function, we have to take $v(z)$ such that

$$e^{v(z)\partial_z} z = f(z)\,. \tag{2.11}$$

From that follows another important relation

$$v(z)\partial_z f(z) = v(f(z))\,, \tag{2.12}$$

which in the mathematical literature is called the Julia equation. The proof is simple:

$$v(z)\partial_z f(z) = v(z)\partial_z e^{v(z)\partial_z} z = e^{v(z)\partial_z} v(z) = v(f(z))\,. \tag{2.13}$$

For completeness and consistency we should be able to show also

$$e^{v(z)\partial_z + dv'(z)} e^{-v(z)\partial_z} = \left[f'(z)\right]^d\,, \tag{2.14}$$

for any d. In order to prove it let us define for $t \in [0,1]$

$$\begin{aligned} f_t(z) &= e^{tv(z)\partial_z} z\,, \\ X_t(z) &= e^{tv(z)\partial_z + dtv'(z)} e^{-tv(z)\partial_z}\,, \end{aligned} \tag{2.15}$$

and derive a differential equation for X_t:

$$\begin{aligned} \partial_t X_t(z) &= dv'(f_t(z)) X_t(z) \\ &= d\frac{\partial_t \partial_z f_t(z)}{\partial_z f_t(z)} X_t(z)\,. \end{aligned} \tag{2.16}$$

Integrating this equation from 0 to t we obtain

$$X_t(z) = \left[f_t'(z)\right]^d\,, \tag{2.17}$$

which for $t = 1$ completes our proof.

For a given analytic $f(z)$ we can formally determine $v(z)$ from (2.11). Plugging the Laurent expansion of f we get all the coefficients v_n recursively. If f vanishes at the origin $f(0) = 0$ and is holomorphic nearby, then only v_n with $n \geq 0$ are nonzero.

An important property of the operators U_f, which follows directly from their definition is

$$U_{f\circ g} = U_f U_g \tag{2.18}$$

for any two functions f and g holomorphic at the origin and obeying $f(0) = g(0) = 0$.[1] For some purposes it is convenient to separate out the global scaling component v_0. This is easily achieved by writing

$$f(z) = f'(0)\frac{f(z)}{f'(0)} \tag{2.19}$$

[1]This condition guarantees, that $f(g(z))$ is holomorphic in a finite neighborhood of the origin. If for example $f(g(z))$ were holomorphic only in some annular region around zero, then there would be a c-number multiplicative anomaly in (2.18) given by $e^{\kappa c}$, where κ is a constant depending on the maps f and g, and c is the central charge of the Virasoro algebra.

and using the composition rule (2.18). It follows that

$$U_f = e^{v_0 L_0} e^{\sum_{n\geq 1} v_n L_n} , \tag{2.20}$$

where

$$\begin{aligned} e^{v_0} &= f'(0) , \\ e^{\sum_{n\geq 1} v_n z^{n+1}\partial_z} &= \frac{f(z)}{f'(0)} . \end{aligned} \tag{2.21}$$

2.2 The definition of wedge states

The wedge states form a subset of more general surface states. Surface state corresponding to a given conformal map $f(z)$ is defined by

$$\langle\, f|\phi\,\rangle = \langle\, f\circ\phi\,\rangle , \qquad \forall\phi \tag{2.22}$$

and therefore

$$\langle f| = \langle 0|U_f . \tag{2.23}$$

Wedge states are defined as a one parameter family of surface states associated to conformal mappings

$$f_r(z) = h^{-1}\left(h(z)^{\frac{2}{r}}\right) = \tan\left(\frac{2}{r}\arctan(z)\right) , \tag{2.24}$$

where $h(z) = (1+iz)/(1-iz)$. The mapping $f_r(z)$ first maps half-disk in the upper half plane into a half-disk in the right half-plane, then shrinks or expands it into a wedge of angle $360^\circ/r$ and finally maps it back into the upper half-plane. We shall denote the wedge states as

$$\langle r| \equiv \langle f_r| = \langle 0|U_{f_r} . \tag{2.25}$$

The associated kets are

$$|r\rangle = U_r^\dagger|0\rangle , \tag{2.26}$$

where $U_r^\dagger$ is defined as BPZ conjugation, which means here that L_n gets replaced by $(-1)^n L_{-n}$.

The one point function (2.22) on the upper half plane can be alternatively calculated on the disk via

$$\langle\, f_r\circ\phi\,\rangle_{\mathrm{half-plane}} = \langle\, F_r\circ\phi\,\rangle_{\mathrm{disk}} , \tag{2.27}$$

where $F_r(z) = h(z)^{2/r}$. From the results of [4] we know that we can apply any conformal transformation, not necessarily $\mathrm{SL}(2,\mathbb{C})$, to any correlator. Since only $\mathrm{SL}(2,\mathbb{C})$ transformations map the complex plane into itself in a single valued manner, a general mapping $f(z)$ will carry the plane into a Riemann surface with branch points. Evaluation of conformal field theory correlators on a general Riemann surface has to be defined, the most natural choice is to evaluate the propagators $\langle\, XX\,\rangle$ and $\langle\, bc\,\rangle$ by mapping them back to the plane. It would seem that we have not gained anything, the bonus comes later when we glue together various pieces of Riemann surfaces.

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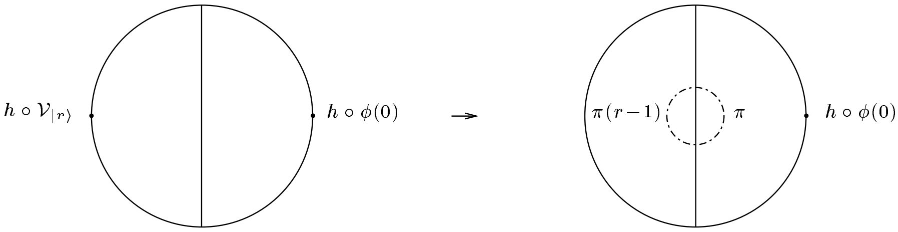

Figure 1: Graphical representation of the BPZ inner product of the wedge state $|r\rangle$ and one auxiliary state $|\phi\rangle$. This can be calculated as a one point function on the Riemann surface with total opening angle πr.

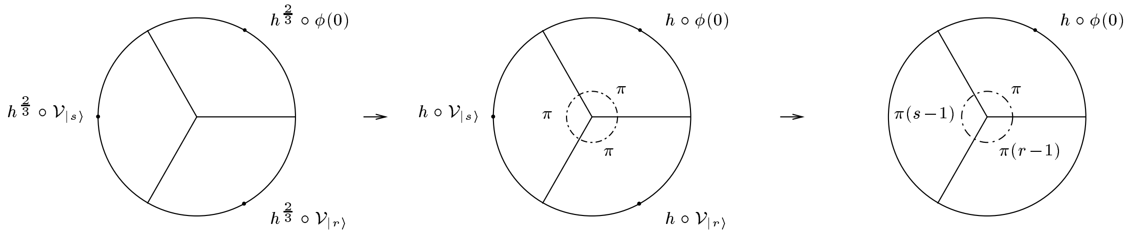

Figure 2: Graphical representation of the 3-vertex contracted with two wedge states $|r\rangle$, $|s\rangle$ and one auxiliary state $|\phi\rangle$. By a conformal mapping we map the disk to a Riemann surface with opening angle 3π and then replace two of the half-disks by helixes of angles $\pi(r-1)$ and $\pi(s-1)$ respectively.

By simple mapping $z \to z^{r/2}$ the disk correlator can be viewed as ordinary one point function on the Riemann surface with total opening angle πr.

BPZ contraction in (2.22) can in general be viewed as two point function on the disk, where at point 1 we insert the vertex operator creating the state ϕ and in -1 the vertex operator for the wedge state. The functional integral over the left and right half-disk separately with fixed boundary condition on the line segment separating them, produces two Schrödinger functionals for these two states. The functional integral over the boundary between the half-disks represents the BPZ contraction itself.

From all of this discussion it should be clear that gluing in half-disk with insertion of the vertex operator for the wedge state (which we do not know explicitly) is equivalent to gluing a piece of Riemann surface of total opening angle $\pi(r-1)$.

The star multiplication of wedge states readily follows (see [17]–[19]). The three vertex contracted with two wedge states $|r\rangle, |s\rangle$ and one auxiliary state $|\phi\rangle$

$$\langle V||r\rangle \otimes |s\rangle \otimes |\phi\rangle \tag{2.28}$$

depicted at figure 2 can be represented first as Riemann surface of total opening angle 3π with three insertions. By the above mentioned equivalence we can replace the half-disks with vertex operators for the wedge states by parts of the Riemann surfaces of angles $\pi(r-1)$ and $\pi(s-1)$. Gluing them together produces a surface with total angle $\pi(r+s-2)$. Equating $r+s-2 = t-1$ gives $t = r+s-1$ and thus the desired composition rule

$$|r\rangle * |s\rangle = |r+s-1\rangle \,. \tag{2.29}$$

Let us give some concrete examples of the wedge states. Using the recursive relations following from (2.21) we have

$$\begin{aligned}
|1\rangle &= e^{L_{-2}-\frac{1}{2}L_{-4}+\frac{1}{2}L_{-6}-\frac{7}{12}L_{-8}+\frac{2}{3}L_{-10}-\frac{13}{20}L_{-12}+\cdots}|0\rangle \\
|2\rangle &= |0\rangle \\
|3\rangle &= e^{-\frac{5}{27}L_{-2}+\frac{13}{486}L_{-4}-\frac{317}{39366}L_{-6}+\frac{715}{236196}L_{-8}-\frac{17870}{14348907}L_{-10}+\cdots}|0\rangle \\
|4\rangle &= e^{-\frac{1}{4}L_{-2}+\frac{1}{32}L_{-4}-\frac{1}{128}L_{-6}+\frac{7}{3072}L_{-8}-\frac{1}{1536}L_{-10}+\cdots}|0\rangle \\
|\infty\rangle &= e^{-\frac{1}{3}L_{-2}+\frac{1}{30}L_{-4}-\frac{11}{1890}L_{-6}+\frac{1}{1260}L_{-8}+\frac{34}{467775}L_{-10}+\cdots}|0\rangle\,.
\end{aligned} \tag{2.30}$$

For general r the wedge state looks as

$$\begin{aligned}
|r\rangle = \exp\Bigg(&-\frac{r^2-4}{3\,r^2}\,L_{-2}+\frac{r^4-16}{30\,r^4}\,L_{-4}-\frac{\left(r^2-4\right)\,\left(176+128\,r^2+11\,r^4\right)}{1890\,r^6}\,L_{-6}+ \\
&+\frac{\left(r^2-4\right)\,\left(r^2+4\right)\,\left(16+32\,r^2+r^4\right)}{1260\,r^8}\,L_{-8}+\cdots\Bigg)|0\rangle\,.
\end{aligned} \tag{2.31}$$

To avoid confusion, the state $|0\rangle$ always denotes the $\mathrm{SL}(2,\mathbb{R})$ invariant vacuum, which is a wedge state $|2\rangle$. Wedge state with $r=0$ simply does not exist. The state $|1\rangle$ is the identity of the star algebra and will be discussed further in section 6. Various aspects of the identity has already been studied in [2, 3], [18] and [21]–[29]. The limiting state $|\infty\rangle$, known as a sliver, is a projector in the star algebra. Since its matter part provides us with a solution to the vacuum string field theory, it has been thoroughly studied in the literature [17, 18] and [30]–[40].

3. Wedge states with insertions

3.1 Basic properties

Let us take a primary field $\mathcal{P}(z)$ of dimension d and a point x inside the unit circle.[2] The wedge states with insertion are defined by

$$\langle f_{r,\mathcal{P},x}| = \langle 0|I\circ\mathcal{P}(x)\,U_{f_r}\,, \tag{3.1}$$

where $Iz=-1/z$. Written as kets they read

$$|f_{r,\mathcal{P},x}\rangle = U_{f_r}^\dagger\,\mathcal{P}(x)|0\rangle\,. \tag{3.2}$$

More generally we can have any number of insertions. From the basic property of conformal field theory on Riemann surfaces

$$\begin{aligned}
\langle\, f_{r,\mathcal{P},x}|\phi\,\rangle &= \langle\, h\circ I\circ\mathcal{P}(x)\;\, h^{2/r}\circ\phi(0)\,\rangle_{\mathrm{disk}} \\
&= \langle\, h^{r/2}\circ I\circ\mathcal{P}(x)\;\, h\circ\phi(0)\,\rangle_{\mathrm{Riemann-surface}}
\end{aligned} \tag{3.3}$$

[2]One may try to go outside of the unit circle by an analytic continuation, but it is quite problematic. Our formulas show clearly that for $x\to\pm i$ the level truncation breaks down, the star product itself is singular. There are two branch cuts starting at $\pm i$ and going to infinity. Across these branch cuts the star product would vary discontinuously and therefore it would fail to be a good product.

we see that the effect of the vertex operator for the wedge state with insertion is again to replace this half-disk with a piece of Riemann surface of total opening angle $\pi(r-1)$ and inserting an operator $\mathcal{P}$ at point

$$h^{\frac{r}{2}} \circ I(x) = e^{ir \arctan x + i(\pi r)/2} \, . \tag{3.4}$$

This equality is actually valid for the standard definition of the function $\arctan x$ in the complex plane. However to appreciate the geometric picture, it is better to temporarily think of x as sitting in the line segment $(-1,1)$ of the real axis. Let us now calculate the star product

$$U_r^\dagger \mathcal{P}_1(x)|0\rangle * U_s^\dagger \mathcal{P}_2(y)|0\rangle \, . \tag{3.5}$$

Again we consider the Witten vertex as Riemann surface obtained by gluing three half-disks, corresponding to the states $U_r^\dagger \mathcal{P}_1(x)|0\rangle, U_s^\dagger \mathcal{P}_2(y)|0\rangle$ and $|\phi\rangle$ in clockwise order. We can replace two of them according to the above rule. Finally we wish to reinterpret this three vertex as a BPZ contraction of ϕ and a wedge state with two insertions. To find the insertion points we have to match simply

$$\begin{aligned} e^{is \arctan y + i(\pi s)/2} &= e^{it \arctan y' + i(\pi t)/2}, \\ e^{ir \arctan x + i(\pi r)/2 + i\pi(s-1)} &= e^{it \arctan x' + i(\pi t)/2} \end{aligned} \tag{3.6}$$

where $t = r + s - 1$. The solution is simply

$$\begin{aligned} x' \equiv g_1(x) &= \cot\left(\frac{r}{t}\left(+\frac{\pi}{2} - \arctan x\right)\right), \\ y' \equiv g_2(y) &= \cot\left(\frac{s}{t}\left(-\frac{\pi}{2} - \arctan y\right)\right), \end{aligned} \tag{3.7}$$

which is manifestly holomorphic in the whole unit disk. Alternatively,[3] one can write these functions as

$$\begin{aligned} x' &= h^{-1}\left(e^{+i\pi(1-r/t)}(h(x))^{r/t}\right), \\ y' &= h^{-1}\left(e^{-i\pi(1-s/t)}(h(y))^{s/t}\right). \end{aligned} \tag{3.8}$$

Having found out the insertion points it is quite simple to work out also the normalization factors coming from the transformation law of the primary fields $\mathcal{P}_1$ and $\mathcal{P}_2$. Altogether, we arrive at

$$U_r^\dagger \mathcal{P}_1(x)|0\rangle * U_s^\dagger \mathcal{P}_2(y)|0\rangle = \left(\frac{r}{t}\cdot\frac{1+x'^2}{1+x^2}\right)^{d_1}\left(\frac{s}{t}\cdot\frac{1+y'^2}{1+y^2}\right)^{d_2} U_{r+s-1}^\dagger \mathcal{P}_1(x')\mathcal{P}_2(y')|0\rangle \tag{3.9}$$

$$= U_{r+s-1}^\dagger \, g_1 \circ \mathcal{P}_1(x) \, g_2 \circ \mathcal{P}_2(y)|0\rangle \tag{3.10}$$

Although we have derived the formula for primary fields, because it was easy to trace the insertion points, the resulting formula (3.10) is actually valid for all local fields and can be used as long as we know the transformation properties of the fields.

[3] I thank I. Ellwood for helpful conversations on this issue.

Let us check this formula on few examples. For the star products with ghost insertions

$$
\begin{aligned}
c(0)|0\rangle * c(0)|0\rangle &= \left(\frac{9}{8}\right)^2 U_3^\dagger\, c\Big(\frac{1}{\sqrt{3}}\Big)\, c\Big(-\frac{1}{\sqrt{3}}\Big)\, |0\rangle\,, \\
|0\rangle * c(0)|0\rangle &= \frac{9}{8} U_3^\dagger\, c\Big(-\frac{1}{\sqrt{3}}\Big)\, |0\rangle\,,
\end{aligned}
\tag{3.11}
$$

we get the same results as obtained previously in [18]. For the star products with energy momentum tensor insertions we find

$$
\begin{aligned}
|0\rangle * T(0)|0\rangle &= \left(\frac{8}{9}\right)^2 U_3^\dagger\, T\Big(-\frac{1}{\sqrt{3}}\Big)\, |0\rangle, \\
T(0)|0\rangle * |0\rangle &= \left(\frac{8}{9}\right)^2 U_3^\dagger\, T\Big(\frac{1}{\sqrt{3}}\Big)\, |0\rangle, \\
T(0)|0\rangle * T(0)|0\rangle &= \left(\frac{8}{9}\right)^4 U_3^\dagger\, T\Big(\frac{1}{\sqrt{3}}\Big)\, T\Big(-\frac{1}{\sqrt{3}}\Big)\, |0\rangle, \\
&= \left(\frac{8}{9}\right)^4 U_3^\dagger\, e^{-(1/\sqrt{3})L_{-1}} T\Big(\frac{2}{\sqrt{3}}\Big)\, T(0)|0\rangle.
\end{aligned}
\tag{3.12}
$$

These formulas can be tested exactly to any given level by explicit calculation of the star products using for instance the conservation laws of [20, 3, 18]. Less trivial example we have tested is

$$
\begin{aligned}
L_{-2}|0\rangle * |\infty\rangle &= \left(\frac{2}{\pi}\right)^4 U'^\dagger_\infty T\left(\frac{2}{\pi}\right) |0\rangle \qquad (3.13)\\
&\simeq 0.164 L_{-2}|0\rangle + 0.105 L_{-3}|0\rangle + 0.067 L_{-4}|0\rangle - 0.055 L_{-2}L_{-2}|0\rangle + \cdots .
\end{aligned}
$$

Here the prime on $U'^\dagger_\infty$ means that the divergent factor $\left(\frac{2}{r}\right)^{L_0}$ has been omitted from its expression. Numerical calculation at level 20 gives result

$$
L_{-2}|0\rangle * |\infty\rangle \simeq 0.180 L_{-2}|0\rangle + 0.110 L_{-3}|0\rangle + 0.067 L_{-4}|0\rangle - 0.058 L_{-2}L_{-2}|0\rangle + \cdots\,, \tag{3.14}
$$

which is in a reasonable agreement.

3.2 Breakdown of the level truncation

Now we would like to use the formula (3.9) to argue that the level truncation calculation breaks down in quite simple cases. Imagine we wish to calculate

$$
L_{-2}|0\rangle * \underbrace{|0\rangle * |0\rangle * |0\rangle * \cdots * |0\rangle}_{(r-1)\text{ times}}\,. \tag{3.15}
$$

Relying on the associativity of the star product, there are many ways to do the calculation. The easiest possibility is to multiply first all the vacua on the right to get wedge state $|r\rangle$ and then to use our formula (3.9) to find

$$
L_{-2}|0\rangle * U_r^\dagger|0\rangle = \left[\frac{r+1}{2}\sin^2\frac{\pi}{r+1}\right]^{-2} U_{r+1}^\dagger T\left(\cot\frac{\pi}{r+1}\right)|0\rangle\,. \tag{3.16}
$$

For general primary fields of dimension d the formula would look the same with the exponent -2 replaced by $-d$. On the other hand we may try to calculate the star product successively, as indicated by the brackets

$$((((L_{-2}|0\rangle * |0\rangle) * |0\rangle) * |0\rangle) * \cdots * |0\rangle)\,. \tag{3.17}$$

This way we get

$$\begin{aligned} L_{-2}|0\rangle * |0\rangle &= \left(\frac{8}{9}\right)^2 U_3^\dagger\, T\!\left(\frac{1}{\sqrt{3}}\right)|0\rangle, \\ (L_{-2}|0\rangle * |0\rangle) * |0\rangle &= U_4^\dagger\, T(1)|0\rangle, \\ ((L_{-2}|0\rangle * |0\rangle) * |0\rangle) * |0\rangle &\stackrel{?}{=} \left[\frac{5}{2}\sin^2\frac{\pi}{5}\right]^{-2} U_5^\dagger\, T\left(\cot\frac{\pi}{5}\right)|0\rangle\,. \end{aligned} \tag{3.18}$$

The right hand side in the last equation follows by straightforward use of the formula (3.9). The reason why we put the question mark above the equality sign is that in reality

$$U_4^\dagger\, T(1)|0\rangle * |0\rangle \tag{3.19}$$

is divergent in the level expansion and cannot be calculated unambiguously. To show that consider

$$U_r^\dagger\, T(z)|0\rangle * |0\rangle \approx \left[\frac{r}{r+1}\, e^{i(\pi/2)(1/(r+1))}\, 2^{-r/(2(r+1))}\, (1+iz)^{-(r+2)/(2(r+1))}\right]^2 U_{r+1}^\dagger\, T(i)|0\rangle \tag{3.20}$$

for $z \to i$. This is clearly divergent in this limit and would be also divergent for any other primary with positive dimension. It means that $U_r^\dagger\, T(z)|0\rangle * |0\rangle$ as a series in z has radius of convergence equal to 1. Therefore the star product $U_4^\dagger\, T(1)|0\rangle * |0\rangle$ is not absolutely convergent in level truncation. The result depends on the order of summation and is thus ambiguous.

3.3 An alternative definition

When we defined the wedge states with insertions in (3.2) the reader could have asked why we did not define them simply as

$$\mathcal{P}(x)\, U_r^\dagger|0\rangle\,. \tag{3.21}$$

This expression as it stands (assuming for a moment that $r \neq 2$) is convergent in the level expansion only for $|x| \geq 1$. To see that, we have to normal order the operator acting on the vacuum to get rid of all positively moded operators annihilating the vacuum. We get

$$\mathcal{P}(x)\, U_r^\dagger|0\rangle = U_r^\dagger\, I \circ f_r \circ I \circ \mathcal{P}(x)|0\rangle \tag{3.22}$$

and we see that in order for the left hand side to be well defined, the series expansion of $I \circ f_r \circ I(x)$ in x has to be convergent.

Now it seems, that we could simply combine the formulas (3.10) and (3.22) to find a star product of states like (3.21). We have to be little careful though. In order to use the formula (3.10) legitimately, one would need $x \in \mathcal{D}_r$, $y \in \mathcal{D}_s$, where

$$\mathcal{D}_r = \{z;\, |I \circ f_r \circ I(z)| < 1\}\,. \tag{3.23}$$

For $r \geq 4$ one can check that $\mathcal{D}_r = \emptyset$. For $2 \leq r < 4$ the domain $\mathcal{D}_r$ is nontrivial, but its intersection with the exterior of the unit circle $|x| \geq 1$ is still empty. Only in the region $1 \leq r < 2$ the intersection is not empty.

Working now in the appropriate region of parameters x, y, r and s we can combine equations (3.10) and (3.22). Assuming further $\mathrm{Re}\, x > 0$ and $\mathrm{Re}\, y < 0$ the resulting formula considerably simplifies

$$\mathcal{P}(x)\, U_r^\dagger|0\rangle * \mathcal{P}(y)\, U_s^\dagger|0\rangle = \mathcal{P}(x)\mathcal{P}(y)\, U_{r+s-1}^\dagger|0\rangle\,, \tag{3.24}$$

which actually remains true for arbitrary $r, s \geq 1$, $\mathrm{Re}\, x > 0$ and $\mathrm{Re}\, y < 0$ by an analytic continuation of the formulas (3.10) and (3.22). It is an obvious manifestation of the fact, that the left part of the first string (region $\mathrm{Re}\, x > 0$) will become the left part of the product, the right part of the second string (region $\mathrm{Re}\, y < 0$) then becomes the right part of the product.

4. Behavior of the wedge state coefficients

Looking at the examples of wedge states (2.30) or on the general formula (2.31) one may wonder whether there is some closed expression for all the coefficients v_n. Another question one may ask, is what is the behavior of v_n for large n. First impression is that for $r = 1$ the coefficients somehow chaotically oscillate between ± 1, whereas for $r > 2$ they decrease exponentially to zero.

To obtain the expressions for the operators U_f with the global scaling component separated we need to solve the equation (2.11) or (2.12) with

$$\tilde{f}_r(z) = \frac{r}{2} h^{-1}\left(h(z)^{2/r}\right) = \frac{r}{2}\tan\left(\frac{2}{r}\arctan(z)\right) \tag{4.1}$$

which satisfies $\tilde{f}_r'(0) = 1$. Given a function f holomorphic in the neighborhood of the origin one can always look for analytic solutions $v(z)$ to the Julia equation (2.12) in terms of formal power series (FPS). The solution is unique up to an overall constant which can be fixed for f of the form

$$f(z) = z + \sum_{n=m}^{\infty} b_n z^n\,, \qquad b_m \neq 0, m \geq 2 \tag{4.2}$$

by requiring

$$v(z) = b_m z^m + \sum_{n=m+1}^{\infty} c_n z^n\,. \tag{4.3}$$

Note that precisely with this normalization the function $v(z)$ satisfies also the (2.11). Such a unique solution is called the *iterative logarithm* and denoted either as f_* or $\mathrm{logit}\, f$. Interesting problem is when this FPS has finite radius of convergence.

It has been proved that if f is a meromorphic function, regular at the origin and having the expansion (4.2) then the formal power series f_* has a positive radius of convergence if and only if

$$f(z) = \frac{z}{1 + bz}, \qquad b \in \mathbb{C}. \tag{4.4}$$

This theorem is implied by the results of I.N. Baker, P. Erdős and E. Jabotinsky, see [41].

Let us see how this result applies to our wedge states. All of the functions $\tilde{f}_r$ are holomorphic near the origin, but only those with $r = \frac{2}{k}$, $k \in \mathbb{Z}$ are meromorphic in the whole complex plane. Apart from the vacuum state $|2\rangle$ all the other $|1\rangle, |2/3\rangle, \ldots$ thus correspond to divergent FPS with zero radius of convergence.

What about the other wedge states? Using the Julia equation and checking the overall normalization one can establish following general properties of the iterative logarithm:

$$\begin{aligned} \operatorname{logit} f &= -\operatorname{logit} f^{-1}\,, \\ \operatorname{logit}(\phi^{-1} \circ f \circ \phi) &= \frac{1}{\phi'}\left((\operatorname{logit} f) \circ \phi\right)\,, \end{aligned} \tag{4.5}$$

where $\phi(z)$ is any analytic function with $\phi(0) = 0$ and $\phi'(0) \neq 0$. From these two relations follows (by taking $\phi(z) = r/2z$)

$$\operatorname{logit} \tilde{f}_{4/r} = -\frac{2}{r} \circ (\operatorname{logit} \tilde{f}_r) \circ \frac{r}{2}\,. \tag{4.6}$$

We thus obtain for the Laurent coefficients of $v^{(r)} = \operatorname{logit} \tilde{f}_r$ important relation

$$v_k^{(4/r)} = -\left(\frac{2}{r}\right)^k v_k^{(r)}, \tag{4.7}$$

which can be readily checked for the explicit expression (2.31). We see that the FPS $\operatorname{logit} \tilde{f}_r$ and $\operatorname{logit} \tilde{f}_{4/r}$ have both either zero or finite radius of convergence simultaneously. Summarizing, the FPS corresponding to the vector field $v(z)$ has zero radius of convergence for all $r = 2/k$ and $r = 2k$ for $k \in \mathbb{Z}, k > 1$. By a limiting procedure this applies in particular to the interesting sliver state $|\infty\rangle$. For $r = 2k + 1$ we cannot simply extend the above argument. One would have to generalize the above quoted mathematical theorem or perhaps rather choose a different method. Nevertheless we expect qualitatively the same kind of behavior.

The absence of any finite radius of convergence means that starting from a certain level, some of the coefficients v_n start to grow faster than any exponential. This rather surprising result is confirmed by the actual calculation of the coefficients up to v_{100} which we have plotted for several wedge states in the appendix B. All the coefficients were calculated exactly using the recursive formula following from (2.11).

To summarize we have shown that the Laurent expansion of $v(z)$ has zero radius of convergence and therefore the function $v(z)$ has an essential singularity at zero. This can be verified independently by studying its poles or zeros. Using the Julia equation one can find an infinite sequence of poles or zeros approaching the origin. Its existence proves that there is an essential singularity. This infinite sequence is easily found numerically, to prove analytically that it is actually infinite seems tedious.

We suspect that the uncontrollable growth of the wedge state coefficients is just an artifact of the symmetrical ordering of the Virasoro generators. It seems likely that for other choices of ordering the coefficients behave much better as was shown for the identity $|I\rangle$ in [25].

If this were not the case, then the series itself could be trusted as asymptotic only. The success of level truncation for the star products of wedge states (see appendix A) would appear to be analogous to the situation in QED, where at first few orders the perturbation theory works perfectly well, but at higher orders it breaks down. From the graphs in the appendix B one can see that for the coefficients up to about v_{20} the coefficients decrease exponentially, this is the basic reason why the low order calculations work well.

Finally let us comment on one technical aspect of the calculation. To calculate e.g. the 100-th derivative at zero of the function $\tilde{f}_r$ for generic r directly, is beyond the capacity of any computer. This problem however can be overcome by the following formula for n odd

$$\frac{d^n}{dx^n} \tan\left(\frac{2}{r}\arctan x\right)|_{x=0} = \sum_{k=1,3,5,\ldots,n} \left(\frac{2}{r}\right)^k \frac{2^{k+1}(2^{k+1}-1)}{(k+1)!} B_{k+1} F(n,k)\,, \tag{4.8}$$

where

$$F(n,k) = \sum_{m_i;\sum m=(n-k)/2} \frac{1}{(2m_1+1)\ldots(2m_k+1)} \tag{4.9}$$

is easily calculable recursively and B_n's are the Bernoulli numbers. For n even the derivative is obviously zero.

5. Wedge states and reparametrizations

We start by deriving new expression for the wedge states which is useful for many explicit analytic calculations. Given the finite conformal transformation

$$f(z) = \tan\left(\frac{2}{r}\arctan z\right) \tag{5.1}$$

we can easily find the vector field $v(z)$ which generates it. From equation (4.5) we get

$$\begin{aligned} v(z) &= \operatorname{logit} f(z) = \operatorname{logit}\left[\arctan^{-1}\circ\frac{2}{r}\arctan(z)\right] = \\ &= \log\frac{2}{r}\cdot(1+z^2)\arctan z\,. \end{aligned} \tag{5.2}$$

The associated finite conformal transformation operator is

$$\begin{aligned} U_r &= \exp\left(\oint\frac{dz}{2\pi i}\,v(z)T(z)\right) \\ &= \exp\left(2\log\frac{r}{2}\left(-\frac{1}{2}L_0 + \sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)(2k+1)}L_{2k}\right)\right). \end{aligned} \tag{5.3}$$

This is a nice expression since it makes manifest the properties (2.18)

$$\begin{aligned} U_r U_s &= U_{rs/2}\,, \\ U_2 &= 1\,. \end{aligned} \tag{5.4}$$

The wedge states are given by

$$|r\rangle = U_r^\dagger|0\rangle = e^{2\log\frac{r}{2}A^\dagger}|0\rangle\,, \tag{5.5}$$

where we denote

$$A = -\frac{1}{2}L_0 + \sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)(2k+1)}L_{2k} \tag{5.6}$$

and $A^\dagger$ is its BPZ conjugate.[4] Calculating the commutator

$$[K_1, A] = \frac{1}{2}K_1 \tag{5.7}$$

we see that all wedge states manifestly obey

$$K_1|r\rangle = 0\,, \tag{5.8}$$

a conservation law first found in [31] and recently discussed in [42]. Next we can calculate the commutator

$$[A, A^\dagger] = -\frac{1}{2}(A + A^\dagger) + \frac{c}{12}\sum_{k=1}^{\infty}\frac{2k}{(2k-1)(2k+1)}\,, \tag{5.9}$$

keeping the central charge c for the time being arbitrary. The last term in the commutator is logarithmically divergent, we could set $c = 0$, but it is interesting to continue with nonzero c assuming some convenient regularization. Now it is useful to introduce a new operator $B = -2A$ with the obvious commutation relation

$$[B, B^\dagger] = B + B^\dagger + \lambda\,, \tag{5.10}$$

where

$$\lambda = \frac{c}{3}\sum_{k=1}^{\infty}\frac{2k}{(2k-1)(2k+1)}\,. \tag{5.11}$$

Then one can easily find

$$BB^{\dagger n}|0\rangle = (B^\dagger + \lambda)\left[\left(B^\dagger + 1\right)^n - B^{\dagger n}\right]|0\rangle \tag{5.12}$$

and

$$B\,e^{\beta B^\dagger}|0\rangle = \left(e^\beta - 1\right)(B^\dagger + \lambda)\,e^{\beta B^\dagger}|0\rangle\,. \tag{5.13}$$

We are interested in particular in

$$X(\alpha, \beta) = e^{\alpha B}e^{\beta B^\dagger}|0\rangle\,, \tag{5.14}$$

which can be found by solving the differential equation

$$\left(\partial_\alpha - \left(e^\beta - 1\right)(\partial_\beta + \lambda)\right)X(\alpha, \beta) = 0\,. \tag{5.15}$$

[4]While this paper was being written, this vector field (without the L_0 part) appeared in a related context in [42].

with the obvious initial condition $X(0,\beta) = e^{\beta B^\dagger}|0\rangle$. The easiest way is to solve first the equation for $\lambda = 0$ and then to recover the correct λ dependence by replacing formally $B^\dagger \to B^\dagger + \lambda$, while keeping B fixed. The solution is thus

$$X(\alpha,\beta) = \left(e^\alpha + e^\beta - e^{\alpha+\beta}\right)^{-\lambda} e^{-\log\left(1-e^\alpha+e^{\alpha-\beta}\right)B^\dagger}|0\rangle\,. \tag{5.16}$$

Having found this general formula we shall now specialize to the wedge states in the combined matter and ghost CFT with vanishing central charge and vanishing anomaly $\lambda = 0$. The separate matter or ghost parts will be treated in subsection 5.2. Applying formula (5.16) for the finite conformal transformation operators (5.3) we get

$$U_r U_s^\dagger|0\rangle = U^\dagger_{2+(2/r)(s-2)}|0\rangle\,. \tag{5.17}$$

As a check, let us note that for the special cases $r = 2$ or $s = 2$ it gives the correct result, less trivially it is also compatible with the composition rule (5.4).

Composition (5.17) can be easily obtained using the gluing theorem.[5] Take an arbitrary state $\langle\phi|$ and calculate both

$$\langle\phi|U_r U_s^\dagger|0\rangle \tag{5.18}$$

and

$$\langle\phi|U_t^\dagger|0\rangle\,, \tag{5.19}$$

The latter is a path integral over a Riemann surface made by gluing a half-disk with ϕ insertion and a piece of helix of total opening angle $\pi(t-1)$. It is a cone of opening angle πt. The former inner product instead, is a path integral over the glued surface of two helices with angles $\pi(r-1)$ and $\pi(s-1)$. The first helix has an insertion of transformed ϕ. After the gluing we transform back by a map $z \to z^{2/r}$ to have a normal insertion of ϕ. So we end up with a integral over a cone with opening angle $(2\pi/r)(r+s-2)$. Matching

$$\frac{2\pi}{r}(r+s-2) = \pi t \tag{5.20}$$

we obtain

$$t = 2 + \frac{2}{r}(s-2)\,, \tag{5.21}$$

which gives precisely the relation we found above quite laboriously using the Virasoro algebra.

Taking the derivative of (5.17) with respect to r at $r = 2$, or just directly from (5.13) we find

$$AU_s^\dagger|0\rangle = \frac{2-s}{s}A^\dagger U_s^\dagger|0\rangle\,. \tag{5.22}$$

We may now introduce an important operator

$$D \equiv A - A^\dagger = \sum_{k=1}^\infty \frac{(-1)^k}{(2k-1)(2k+1)}(L_{2k} - L_{-2k})\,, \tag{5.23}$$

[5] I would like to thank A. Sen for this suggestion.

which acts as an exterior derivative of the star product algebra because it is a linear combination of the operators $K_{2k} = L_{2k} - L_{-2k}$ [3, 18]. Its action on a wedge state is

$$DU_s^\dagger|0\rangle = \left(\frac{2}{s} - 2\right) A^\dagger U_s^\dagger|0\rangle \,. \tag{5.24}$$

It obviously vanishes for the identity wedge state with $s = 1$, but it also annihilates the sliver with $s = \infty$, since

$$A^\dagger U_\infty^\dagger|0\rangle = 0 \tag{5.25}$$

is just the L_0 conservation law for the sliver [31].

By the same method as we used to get (5.17) we can also calculate

$$e^{-\alpha D} U_s^\dagger|0\rangle = U_{1+e^\alpha(s-1)}^\dagger|0\rangle \,. \tag{5.26}$$

It says, that starting with any regular wedge state $|s\rangle$, where $1 < s < \infty$ we can obtain any other wedge state by a finite reparametrization $e^{-\alpha D}$. In the limit $\alpha \to -\infty$ we then recover the identity wedge state $|I\rangle$ and in the other limit $\alpha \to +\infty$ we get the sliver. The conclusion is that the identity and the sliver are just singular reparametrizations of the ordinary vacuum state. This is in complete agreement with the geometric arguments of [16].

To end up this discussion we would like to note that (5.17) and (5.26) can be written in more generality as operator statements

$$\begin{aligned} U_r U_s^\dagger &= U_{2+(2/r)(s-2)}^\dagger U_{2+(2/s)(r-2)}, \\ e^{-\alpha D} &= U_{1+e^\alpha}^\dagger U_{1+e^{-\alpha}} \,. \end{aligned} \tag{5.27}$$

It follows by considering the BPZ or hermitian conjugation and from the fact that the right hand side should be expressible entirely in terms of A and $A^\dagger$.

5.1 Star products of wedge states without gluing theorem

From the definition of the star product one can easily obtain formulas for star product of vacuum state with any other state from the Fock space

$$\begin{aligned} |0\rangle * |\psi\rangle &= U_3^\dagger e^{-(1/\sqrt{3})L_{-1}} \left(\frac{4}{3}\right)^{L_0} e^{-(1/\sqrt{3})L_1} U_3|\psi\rangle \,, \\ |\psi\rangle * |0\rangle &= U_3^\dagger e^{(1/\sqrt{3})L_{-1}} \left(\frac{4}{3}\right)^{L_0} e^{(1/\sqrt{3})L_1} U_3|\psi\rangle \,. \end{aligned} \tag{5.28}$$

These formulas make perfectly sense in the level expansion, since if $|\psi\rangle$ contains finitely many levels, the whole expression can be calculated to any given level exactly, in finitely many steps. We have also checked it independently on several examples.

For the applications to the wedge states it is however convenient to rewrite them using the formulas

$$\begin{aligned} e^{\alpha K_1} &= e^{\tan\alpha\, L_{-1}} \,(\cos\alpha)^{-2L_0}\, e^{\tan\alpha\, L_1}, \\ e^{\alpha K_1} e^{\beta A} &= e^{\beta A} e^{\alpha\, e^{(\beta/2)} K_1}, \end{aligned} \tag{5.29}$$

to get

$$\begin{aligned}|0\rangle * |\psi\rangle &= U_3^\dagger e^{-(\pi/6)K_1} U_3|\psi\rangle = U_3^\dagger U_3 e^{-(\pi/4)K_1}|\psi\rangle\,,\\ |\psi\rangle * |0\rangle &= U_3^\dagger e^{+(\pi/6)K_1} U_3|\psi\rangle = U_3^\dagger U_3 e^{+(\pi/4)K_1}|\psi\rangle\,.\end{aligned} \tag{5.30}$$

From (5.7), (5.8) and (5.17) it follows that [6]

$$\begin{aligned}|0\rangle * |r\rangle &= |r+1\rangle\,,\\ |r\rangle * |0\rangle &= |r+1\rangle\,.\end{aligned} \tag{5.31}$$

Star product of two general wedge states can be calculated using the formula (5.26) and using the property

$$e^{-\alpha D}\left(|\phi\rangle * |\chi\rangle\right) = e^{-\alpha D}|\phi\rangle * e^{-\alpha D}|\chi\rangle \qquad \forall\,\phi,\chi\,, \tag{5.32}$$

valid for any derivative D of the star product. Writing the wedge states as

$$\begin{aligned}|r\rangle &= e^{-\log(r-1)\,D}|0\rangle,\\ |s\rangle &= e^{-\log(r-1)\,D}\left|1+\frac{s-1}{r-1}\right\rangle,\end{aligned} \tag{5.33}$$

we easily find

$$\begin{aligned}|r\rangle * |s\rangle &= e^{-\log(r-1)\,D}\left|2+\frac{s-1}{r-1}\right\rangle = \\ &= |r+s-1\rangle\,.\end{aligned} \tag{5.34}$$

5.2 On the matter and ghost parts of wedge states

Since all the wedge states are exponentials of total Virasoro operators, they are naturally factorized into a matter and ghost parts. What are the properties of these parts? What happens when the matter part corresponds to a different wedge state than the ghost part? We will see that our techniques can be used to get some insight into these questions.

By formal replacement $A \to A - \lambda/4$ and $A^\dagger \to A^\dagger - \lambda/4$ in (5.27) we can derive important relations valid in a CFT with nonzero central charge c

$$\begin{aligned}U_r U_s^\dagger &= \left(\frac{rs}{2(r+s-2)}\right)^{\lambda} U^\dagger_{2+(2/r)(s-2)} U_{2+(2/s)(r-2)},\\ e^{-\alpha D} &= \left(\cosh\frac{\alpha}{2}\right)^{-\lambda} U^\dagger_{1+e^{\alpha}} U_{1+e^{-\alpha}},\end{aligned} \tag{5.35}$$

and the analog of (5.26) is

$$e^{-\alpha D} U_s^\dagger|0\rangle = \left[\frac{1+e^{\alpha}(s-1)}{s e^{\alpha/2}}\right]^{-\lambda} U^\dagger_{1+e^{\alpha}(s-1)}|0\rangle\,. \tag{5.36}$$

[6] Note that in our notation $|0\rangle \equiv |2\rangle$.

How can we now calculate the star product? Let us start with the simplest one $|0\rangle * |r\rangle$. Using the explicit formula (5.30) we get

$$|0\rangle * |r\rangle = \left(\frac{3r}{2(r+1)}\right)^{\lambda} |r+1\rangle \,. \tag{5.37}$$

Note that this result is consistent with cyclicity

$$\langle\, A, B * C\,\rangle = \langle\, B, C * A\,\rangle = \langle\, C, A * B\,\rangle \,, \tag{5.38}$$

since applying $\langle 0|$ to the left hand side of (5.37) we find using (5.35)

$$\langle 0|(|0\rangle * |r\rangle) = \langle\, r|3\,\rangle = \left(\frac{3r}{2(r+1)}\right)^{\lambda} \,, \tag{5.39}$$

the same as by applying it to the right hand side.

To calculate star product of arbitrary two wedge states we start from

$$|r\rangle * |s\rangle = e^{\mathcal{A}_w} |r+s-1\rangle \,, \tag{5.40}$$

where $\mathcal{A}_w$ is some anomaly in this composition rule. That the product should take this form in general we know from the factorization of the three vertex and from the composition of wedges in combined matter and ghost CFT with $c = 0$. The anomaly can again be determined by contracting the equation with the vacuum $\langle 0|$, using the cyclicity (5.38) and equations (5.37) (5.35). The result is

$$e^{\mathcal{A}_w} = \left(\frac{3rs}{4(r+s-1)}\right)^{\lambda} \,. \tag{5.41}$$

Combining now the wedge states in the matter and ghost CFT into $|r, \tilde{r}\rangle = |r\rangle_m \otimes |\tilde{r}\rangle_{gh}$ we find

$$|r, \tilde{r}\rangle * |s, \tilde{s}\rangle = \left(\frac{3rs}{4(r+s-1)}\right)^{\lambda} \left(\frac{3\tilde{r}\tilde{s}}{4(\tilde{r}+\tilde{s}-1)}\right)^{\tilde{\lambda}} |r+s-1, \tilde{r}+\tilde{s}-1\rangle \,. \tag{5.42}$$

Note, that we allow for arbitrary central charges and therefore anomalies $\lambda, \tilde{\lambda}$ in both CFT's. This is important for applications to the twisted ghost CFT in [16].

The anomalous factors in (5.42) may look at first rather odd, but in fact they are nicely compatible with commutativity and associativity.

Now let us study what happens under finite or infinitesimal reparametrizations generated by

$$D^X = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)(2k+1)} K^X_{2k} \,, \tag{5.43}$$

where the superscript X refers to matter part of the CFT. On general grounds of cyclicity one can argue that the anomaly $\mathcal{A}$ appearing in

$$D^X |\phi\rangle * |\chi\rangle + |\phi\rangle * D^X |\chi\rangle = (\mathcal{A} + D^X)\,(|\phi\rangle * |\chi\rangle) \,, \tag{5.44}$$

is the same for all the unbalanced wedge states. The integrated form is then

$$e^{-\alpha D^X}|\phi\rangle * e^{-\alpha D^X}|\chi\rangle = e^{-\alpha\mathcal{A}}e^{-\alpha D^X}\left(|\phi\rangle * |\chi\rangle\right). \tag{5.45}$$

Since we know explicitly from (5.36) that

$$e^{-\alpha D^X}|r,\tilde{r}\rangle = \left[\frac{1+e^{\alpha}(r-1)}{re^{\alpha/2}}\right]^{-\lambda}|1+e^{\alpha}(r-1),\tilde{r}\rangle \tag{5.46}$$

we may calculate both sides of (5.45) to find the anomaly

$$\mathcal{A} = -\frac{\lambda}{2}. \tag{5.47}$$

An alternative procedure is to calculate the anomaly directly by summing the anomalies associated to individual K_{2k}^X calculated in [3, 24, 18]

$$\mathcal{A} = \sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)(2k+1)}\cdot(-3)\frac{5c}{54}\cdot k(-1)^k = -\frac{5}{12}\lambda. \tag{5.48}$$

Although the numbers $-\frac{1}{2}$ and $-\frac{5}{12}$ are quite close, we got a clear clash between these two equally justifiable methods.[7] We have to conclude that our basic assumption of the cyclicity is wrong. If we include the unbalanced wedge states into the string field algebra and/or we allow for reparametrization generated by D^X we definitely violate the cyclicity of the three vertex.

Let us end this section with another remark. From relation (5.35) follows that the norm squared of the matter part of wedge states is

$$\langle\, r|r\,\rangle = \left(\frac{r^2}{4(r-1)}\right)^{\lambda}. \tag{5.49}$$

It is divergent for all r. In particular the matter part of the state $|3\rangle$, which is a star product of two vacuum states has infinite norm. This was recently found also in [37]. The norm of the identity $r=1$ and of the sliver $r=\infty$ is even more divergent, it diverges even at finite λ.

6. Identity string field

In this section we would like to turn our attention to the identity element of the string field algebra. In general, identity element of any algebra is quite an important object. It may or may not exist. For the string field star algebra we shall give an explicit construction bellow. However, since we are lacking a mathematically satisfactory definition of the algebra itself we cannot say whether the identity actually belongs to the space or not. Good definition of the algebra would be to require having only finite norm states, for instance, but this only shifts the problem to finding a good norm. The canonical norm defined by the hermitian

[7]Further clash arises when we want to consistently include the identity $|1,1\rangle$. This would require $\mathcal{A} = -(3/4)\lambda$.

inner product does not work, since as we have seen just above in section 5.2 even the product of two vacuum states $|0\rangle$ does not have finite norm.

Let us now forget about the problems whether the identity should belong to the algebra or not and let us describe its various forms. In the Witten's formulation where the star product of two string fields in the Schrödinger representation is defined by (ignoring ghosts for the moment)

$$(\Psi_1 * \Psi_2)(X_0(\sigma)) = \int \mathcal{D}X_1(\sigma)\mathcal{D}X_2(\sigma)\, \Psi_1(X_1(\sigma))\Psi_2(X_2(\sigma)) \times \\ \times \prod_{0\le\sigma\le\pi/2} \delta(X_1(\sigma)-X_0(\sigma))\delta(X_2(\sigma)-X_1(\pi-\sigma))\delta(X_0(\pi-\sigma)-X_2(\pi-\sigma))\,, \tag{6.1}$$

the identity is clearly the functional

$$\langle\, X(\sigma)|I\,\rangle = \prod_{0\le\sigma\le\pi/2} \delta(X(\sigma) - X(\pi-\sigma))\,. \tag{6.2}$$

To write it in the Fock space, first one need to use the mode expansion $X(\sigma) = x_0 + \sqrt{2}\sum_{n=1}^{\infty} x_n \cos n\sigma$ to get

$$\langle\, X(\sigma)|I\,\rangle = \prod_{n=1,3,\dots} \delta(x_n)\,. \tag{6.3}$$

Then after expressing the coherent states $|x_n\rangle$ using the creation operators we find by a simple calculation

$$|I\rangle = \int \mathcal{D}X(\sigma)\,|X(\sigma)\rangle\langle\, X(\sigma)|I\,\rangle \\ = e^{-(1/2)\sum_{n=0}^{\infty}(-1)^n a_n^\dagger a_n^\dagger}|0\rangle\,. \tag{6.4}$$

Treating carefully the ghosts one gets in the oscillator approach [3]

$$|I\rangle = \frac{i}{4} b(i)b(-i)e^{\sum_{n=1}^{\infty}(-1)^n(-(1/2)a_n^\dagger a_n^\dagger + c_{-n}b_{-n})}c_1c_0|0\rangle\,. \tag{6.5}$$

From the geometrical representation of the star product discussed in section 2 it is clear that the identity is the wedge state $|1\rangle$

$$|I\rangle = e^{L_{-2}-(1/2)L_{-4}+(1/2)L_{-6}-(7/12)L_{-8}+(2/3)L_{-10}-(13/20)L_{-12}+\cdots}|0\rangle\,. \tag{6.6}$$

We have calculated the higher level terms in the exponent exactly up to L_{-100} term, the results are plotted in the graph figure 3. It is quite surprising that up to the level 20 the coefficients are less or around one, but then start to diverge faster than any exponential. This divergence should be however viewed as some combinatorial divergence related to unfortunate ordering of the Virasoro generators. Indeed, a nice alternative form of the identity has been found by Ellwood et. al. [25]

$$|I\rangle = \left(\prod_{n=2}^{\infty} e^{-(2/2^n)L_{-2^n}}\right) e^{L_{-2}}|0\rangle\,, \tag{6.7}$$

in which higher level terms have manifestly well behaved coefficients.

Finally let us note, that one can easily perform an explicit calculation in level truncation to show, that various forms of the identity (6.5), (6.6) and (6.7) are indeed in mutual agreement.

6.1 Conservation laws for the identity

Virasoro conservation laws. Let us recall the derivation of the conservation laws due to Rastelli and Zwiebach [18]. We start with a global coordinate $\tilde{z}$ on the 1-punctured disk, associated to the identity $\langle I|$. For any holomorphic vector field $\tilde{v}(\tilde{z})$ we have the basic identity

$$\langle I| \oint_{\mathcal{C}} d\tilde{z}\, \tilde{v}(\tilde{z})\tilde{T}(\tilde{z}) = 0\,, \tag{6.8}$$

where $\mathcal{C}$ is a contour encircling the puncture. Passing to the local coordinate z around the puncture we get

$$\langle I| \oint_{\mathcal{C}} dz\, v(z) \left(T(z) - \frac{c}{12} S(f^{360^\circ}(z), z)\right) = 0\,, \tag{6.9}$$

where

$$S(f^{360^\circ}(z), z) = 6(1+z^2)^{-2} = 6\sum_{m=1}^{\infty} m(-1)^{m-1} z^{2(m-1)} \tag{6.10}$$

is the schwarzian derivative reflecting the non-tensor character of the energy momentum tensor when the central charge c is nonzero. For a particular choice of the vector field $v(z) = z^{n+1} - (-1)^n z^{-n+1}$, which is holomorphic everywhere in the global coordinate $\tilde{z}$ except the puncture, we get

$$\begin{aligned} K_{2n}|I\rangle &= -\frac{c}{2} n(-1)^n |I\rangle\,, \\ K_{2n+1}|I\rangle &= 0\,, \end{aligned} \tag{6.11}$$

where we define

$$K_n = L_n - (-1)^n L_{-n}\,. \tag{6.12}$$

The same identities can be derived for the b ghost, in that case, there is no anomaly however.

Let us further comment on some applications of the formulas we have obtained. First one can rewrite the state $T(z)|I\rangle$ in a form which is manifestly well defined in the level expansion and perform the geometric sums provided $|z| > 1$.

$$T(z)|I\rangle = \frac{c}{2}\frac{1}{(1+z^2)^2}|I\rangle + \frac{1}{z^2}L_0|I\rangle + \frac{1}{z^2}\sum_{n\geq 1}\left(z^n + (-1)^n z^{-n}\right) L_{-n}|I\rangle\,. \tag{6.13}$$

From these identities, and those for the b ghost, one can easily check the overlap equations

$$\begin{aligned} \left(T(z) - \frac{1}{z^4} T\left(-\frac{1}{z}\right)\right)|I\rangle &= 0\,, \\ \left(b(z) - \frac{1}{z^4} b\left(-\frac{1}{z}\right)\right)|I\rangle &= 0\,. \end{aligned} \tag{6.14}$$

Conservation of the c-ghost. We start from the identity

$$\langle I| \oint_{\mathcal{C}} dz\, \phi(z) c(z) = 0\,, \tag{6.15}$$

where $\phi(z)$ is a quadratic differential holomorphic everywhere except at the puncture located at the origin, and $\mathcal{C}$ is a contour encircling the puncture. The $\phi(z)$ transforms as follows

$$\tilde{\phi}(\tilde{z}) = \left(\frac{dz}{d\tilde{z}}\right)^2 \phi(z)\,. \tag{6.16}$$

We shall pass from the local coordinate z around the puncture to the global coordinate on the 1-punctured disk

$$\tilde{z} = \frac{2z}{1-z^2}\,. \tag{6.17}$$

For the particular choice of the quadratic differentials

$$\begin{aligned} \phi_{2n}(z) &= \frac{1}{z^2}\left(z^n - \left(-\frac{1}{z}\right)^n\right)^2 , \\ \phi_{2n+1}(z) &= \frac{1}{z^2}\left(z^{2n+1} - \left(-\frac{1}{z}\right)^{2n+1} - (-1)^n\left(z-\frac{1}{z}\right)\right), \end{aligned} \tag{6.18}$$

where $n \geq 1$, the transformed differentials are

$$\tilde{\phi}_{2n}(\tilde{z}) = \frac{4}{\tilde{z}^{2n+2}}\left(\sum_{k=1,3,5,\ldots}\binom{n}{k}\left(1+\tilde{z}^2\right)^{\frac{k-1}{2}}\right)^2 , \tag{6.19}$$

$$\tilde{\phi}_{2n+1}(\tilde{z}) = -\frac{2}{\tilde{z}^{2n+3}}\frac{1}{1+\tilde{z}^2}\left(-(-1)^n\tilde{z}^{2n} + \sum_{k=0,2,4,\ldots}\binom{2n+1}{k}\left(1+\tilde{z}^2\right)^{k/2}\right).$$

All the sums here are finite due to the combinatorial factors which are defined to be zero whenever the lower entry is bigger than the upper entry.

The quadratic differentials expressed in the global coordinate system $\tilde{z}$ are holomorphic in the whole complex plane except zero, in particular they do not have any singularity at $\pm i$. Therefore one may derive from (6.15) the conservation laws[8]

$$\begin{aligned} C_{2n}|I\rangle &= (-1)^n C_0|I\rangle\,, \\ C_{2n+1}|I\rangle &= (-1)^n C_1|I\rangle\,, \end{aligned} \tag{6.20}$$

where in general we define

$$C_k = c_k + (-1)^k c_{-k}\,. \tag{6.21}$$

Let us remark that a naive conservation law based on the quadratic differential

$$\phi(z) = \frac{1}{z^2}\left(z^n - \left(-\frac{1}{z}\right)^n\right) \tag{6.22}$$

fails, since $\tilde{\phi}(\tilde{z})$ does have poles in $\pm i$. This is actually a simple manifestation of the midpoint anomalies.

[8]For a derivation using oscillators see [27].

Again, as we have done for the energy momentum tensor, we can rewrite the state $c(z)|I\rangle$ in a form which is manifestly well defined in the level expansion provided $|z|>1$,

$$c(z)|I\rangle = -z\frac{1-z^2}{1+z^2}\,c_0|I\rangle + \frac{z^2}{1+z^2}\left(c_1 - c_{-1}\right)|I\rangle + z\sum_{n\geq 1}\left(z^n - \left(-\frac{1}{z}\right)^n\right)c_{-n}|I\rangle\,. \tag{6.23}$$

The single poles for $z\to\pm i$ were first found by other means by [22, 23]. From this formula, it is a simple exercise to verify the overlap equation

$$\left(c(z) - z^2 c\left(-\frac{1}{z}\right)\right)|I\rangle = 0\,. \tag{6.24}$$

Another observation we can make is about $c_L|I\rangle$, where

$$c_L = \frac{1}{2}c_0 + \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\left(c_{2k+1} + c_{-(2k+1)}\right)\,. \tag{6.25}$$

We can easily calculate that

$$c_L|I\rangle = \frac{1}{2}c_0|I\rangle + \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}c_{-(2k+1)}|I\rangle + \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}C_1|I\rangle\,, \tag{6.26}$$

which is divergent due to the last term. This fact rules out the possibility of relating solutions to the vacuum and Witten's cubic string field theories through the most naive $(c_L - Q_L)|I\rangle$ shift. For more sophisticated possibilities see [29].

Current conservation laws. For completeness let us also consider conservation laws for currents. Let us take in general a holomorphic current $J(z)$ having the following OPE with the stress tensor

$$T(z)J(0) = \frac{2q}{z^3} + \frac{J(0)}{z^2} + \frac{\partial J(0)}{z}\,. \tag{6.27}$$

Under finite conformal transformations it transforms as

$$\frac{dw}{dz}\tilde{J}(w) = J(z) - q\frac{d^2w}{dz^2}\left(\frac{dw}{dz}\right)^{-1}\,. \tag{6.28}$$

The anomalous constant q is zero for the BRST current and ∂X currents, for the ghost number current it is $-3/2$. Following the same procedure as above, one may derive conservation laws

$$\begin{aligned} H_0|I\rangle &= 0\,,\\ H_{2k+1}|I\rangle &= 0\,,\\ H_{2k}|I\rangle &= (-1)^k 2q|I\rangle\,, \end{aligned} \tag{6.29}$$

where

$$H_k = J_k + (-1)^k J_{-k}\,. \tag{6.30}$$

6.2 Anomalous properties of the identity

One particularly puzzling aspect [18, 29] of the identity is the following: We know that c_0 acts as a derivation on the string field algebra. Therefore we can write formally

$$c_0|\Psi\rangle = c_0\left(|I\rangle * |\Psi\rangle\right) = c_0|I\rangle * |\Psi\rangle + |I\rangle * c_0|\Psi\rangle = c_0|I\rangle * |\Psi\rangle + c_0|\Psi\rangle\,,$$

from which follows that

$$c_0|I\rangle * |\Psi\rangle = 0 \tag{6.31}$$

for any string field $|\Psi\rangle$. Naively one would conclude (taking $|\Psi\rangle = |I\rangle$) that $c_0|I\rangle = 0$, but that is manifestly not true. One could possibly imagine several ways out:

- $|I\rangle$ is not a true identity on all states;
- c_0 is not a true derivation on the whole algebra ;
- even though $c_0|I\rangle \neq 0$, still we have $c_0|I\rangle * |\Psi\rangle = 0$ for any "well behaved" state $|\Psi\rangle$;
- simply $c_0|I\rangle * |\Psi\rangle$ is not well defined in the level expansion .

We will argue for the last possibility, but in some limited sense all the explanations are true.

The derivations of the fact that $|I\rangle$ is the identity and that c_0 is a derivation on the algebra are quite firm when one restricts on well behaved states. Again it is difficult to say what is a well defined state, but those which contain finitely many levels certainly are. We have checked numerically that the identity is an identity on many states, it seems that it is an identity even on itself and other wedge states, which might be otherwise somehow problematic.

To check the third possibility it is best to look first at some example. Let us calculate

$$c_0|I\rangle * |0\rangle\,.$$

In general there are many ways to do the calculation. The most naive way would be to truncate the identity to some maximal level, legally use the c_0 conservation to arrive at $c_0(|I\rangle * |0\rangle) - |I\rangle * c_0|0\rangle$ which is indeed very close to zero, since the identity $|I\rangle$ works well for the states $|0\rangle, c_0|0\rangle$. A sort of "canonical" way of calculation suggested in [18] is to re-order $c_0|I\rangle$ to have only the ghost c_1 acting on the vacuum and Virasoro generators acting on it from the left. Then one can use the recursive relations of [3, 20, 18] for the Virasoro generators, to reduce the expression to the linear combination of terms

$$L_{-a}L_{-b}\ldots(c_1|0\rangle * |0\rangle)\,.$$

Actually we can perform this calculation exactly even with some sort of regularization. As a first step let us commute the c-ghost to the vacuum. In more generality, we will do it for an arbitrary wedge state instead of the identity and for convenience work with the bra vectors

$$\langle r|c_0 = \langle 0|U_r c_0 = \langle 0|\left(U_r c_0 U_r^{-1}\right)U_r\,. \tag{6.32}$$

The factor in the bracket is readily

$$
\begin{aligned}
U_r c_0 U_r^{-1} &= \oint \frac{dz}{2\pi i}\frac{1}{z^2} U_r c(z) U_r^{-1} \\
&= -4r^2 \sum_n c_n \oint \frac{dw}{2\pi i}\frac{1}{w^{n-1}}\frac{(1+w^2)^{r-2}}{[(1+iw)^r-(1-iw)^r]^2} \\
&= c_0 + \frac{r^2-4}{3}c_2 + \frac{r^4-10r^2+24}{15}c_4 + \frac{10r^6-168r^4+945r^2-1732}{945}c_6 + \cdots .
\end{aligned}
\tag{6.33}
$$

For general r it looks difficult to find a closed expression, for the identity state which corresponds to $r=1$ the sum can be easily performed

$$U_1 c_0 U_1^{-1} = -\frac{i}{2}\left(c(i)-c(-i)\right) . \tag{6.34}$$

Therefore

$$c_0|I\rangle = \frac{i}{2} U_1^\dagger \left(c(i)-c(-i)\right)|0\rangle . \tag{6.35}$$

This is already in the form for which we know how to calculate the star products exactly. Recall however that the insertion points $\pm i$ are singular. A natural regularization from the point of view of level expansion is to replace

$$c(i)-c(-i) \quad \longrightarrow \quad c(ai)-c(-ai)\,, \tag{6.36}$$

where $a<1$ is approaching unity from bellow.[9] Let us calculate in more generality

$$
\begin{aligned}
\frac{i}{2}U_1^\dagger\left(c(ia)-c(-ia)\right)|0\rangle * U_s^\dagger|0\rangle &= \frac{is}{2}\frac{1-a^2}{1+x'^2(ia)}U_s^\dagger c(x'(ia))|0\rangle - \\
&\quad -\frac{is}{2}\frac{1-a^2}{1+x'^2(-ia)}U_s^\dagger c(x'(-ia))|0\rangle\,,
\end{aligned}
\tag{6.37}
$$

where

$$x'(ia) = -i\frac{(a-1)^{2/s}+(a+1)^{2/s}}{(a-1)^{2/s}-(a+1)^{2/s}} . \tag{6.38}$$

This has well defined limit for $a\to\pm 1$, it is either i or $-i$ respectively. The prefactors require little bit more care, since they are limits of 0/0 type.

For $s>2$, the whole expression is well defined and we get

$$\lim_{a\to 1^-}\frac{i}{2}U_1^\dagger\left(c(ia)-c(-ia)\right)|0\rangle * U_s^\dagger|0\rangle = 0\,. \tag{6.39}$$

For $s=2$ we get

$$\lim_{a\to 1^-}\frac{i}{2}U_1^\dagger\left(c(ia)-c(-ia)\right)|0\rangle * |0\rangle = i(c(i)-c(-i))|0\rangle\,. \tag{6.40}$$

[9] Another interesting regularization would be to keep $r\neq 1$, but that is technically rather cumbersome, due to the presence of nontrivial contour integral in (6.33). Yet another possibility is to replace U_1 with U_r for $r\neq 1$ in (6.37), this is however an almost trivial modification of our calculations.

For $s < 2$ at least one of the two prefactors in (6.37) is divergent. The conclusion of the above calculation is that the result of the calculation $c_0|I\rangle * |0\rangle$ is highly sensitive to the method used. From the mathematical point of view, $c_0|I\rangle$ does not belong to the star algebra. If we want to have the derivation c_0 defined on all of the algebra, we should conclude, that neither the identity $|I\rangle$ belongs to the algebra. Alternatively we can think of the identity belonging to the algebra, but then the derivative c_0 maps some elements of the algebra out of it.

One could imagine other ways how to study the object $c_0|I\rangle$. We would like to warn the reader of the following problem

$$\left[\oint \frac{dz}{2\pi i}\frac{c(z)}{z^2}\right]|I\rangle = c_0|I\rangle\,,$$
$$\oint \frac{dz}{2\pi i}\left[\frac{c(z)}{z^2}|I\rangle\right] = -c_0|I\rangle\,, \tag{6.41}$$

where the brackets in the second case mean, that we are evaluating the expression, i.e. calculating it in the level expansion through formula (6.23). Both integrals are along small contours around the origin. To get identical results, we would need to include the points $\pm i$ inside the contour of integration. The problem can be traced back to the fact, that to make sense of $c(z)|I\rangle$ without analytic continuation, we need to remain outside of the unit circle.

Finally we would like to address the issue of the "integrated" anomaly. Let us consider

$$\langle V_{123}|(c_0^{(1)} + c_0^{(2)} + c_0^{(3)})|I\rangle \otimes |\Psi_2\rangle \otimes |\Psi_3\rangle\,, \tag{6.42}$$

where for simplicity $|\Psi_{2,3}\rangle$ are two ghost number one string fields. This is equal to

$$\langle \mathrm{bpz}\,\Psi_3|\Big(c_0|I\rangle * |\Psi_2\rangle + |I\rangle * c_0|\Psi_2\rangle - c_0\left(|I\rangle * |\Psi_2\rangle\right)\Big) = \langle \mathrm{bpz}\,\Psi_3|\Big(c_0|I\rangle * |\Psi_2\rangle\Big)\,, \tag{6.43}$$

which we call "integrated" anomaly. In level truncation we can safely use the cyclicity to get further

$$-\langle I|c_0\Big(|\Psi_2\rangle * |\Psi_3\rangle\Big) = \langle\, \mathrm{bpz}\,\Psi_3|c_0|\Psi_2\,\rangle - \langle\, \mathrm{bpz}\,c_0\,\Psi_3|\Psi_2\,\rangle = 0\,. \tag{6.44}$$

This is as it should be, since our starting expression (6.42) is also manifestly zero in the level expansion. The moral is that while $c_0|I\rangle * |\Psi_2\rangle$ itself is ill defined, its BPZ inner products with well behaved states can be consistently set to zero.

6.3 Application to the tachyon condensation

There is one remarkably simple application of the above results to the study of tachyon condensation in ordinary cubic string field theory. The string field action at the critical point should be invariant under a general variation

$$\delta S = -\frac{1}{g^2}\Big[\langle\, \delta\Phi|Q\Phi_0\,\rangle + \langle\, \delta\Phi|\Phi_0 * \Phi_0\,\rangle\Big] = 0\,. \tag{6.45}$$

Level:	0	2	4	6	8
$\langle I\|C_1Q\|\Phi_0\rangle$	0.3294	−0.1684	0.1303	−0.0422	0.0671

Table 1: Values of $\langle I|C_1Q|\Phi_0\rangle$ calculated at different levels of truncation. Ideally it should vanish as the level is increased.

Taking a variation of the form $\delta\Phi = C_1|I\rangle$ we see, that the variation of the cubic term vanishes since C_1 like c_0 is a derivation of the star algebra and because the identity commutes with everything. Therefore we are led to the conclusion that the tachyon condensate should satisfy

$$\langle I|C_1Q|\Phi_0\rangle = 0\,. \tag{6.46}$$

Let us see how well this works in the level truncation. We start our analysis with a level 10 numerical solution Φ_0 of the equations of motion in Siegel gauge which has been kindly provided to us by Gaiotto, Rastelli, Sen and Zwiebach.[10] Now with this solution (treating it as an exact solution), we calculate level $n = 0, 2, 4, 6, 8$ approximation to the expression $\langle I|C_1Q|\Phi_0\rangle$ by truncating both the identity $\langle I|$ and $C_1Q|\Phi_0\rangle$ to that level. We cannot go to level 10 in this test, since this would require knowledge of level 12 terms in the solution. The results are listed in the table 1. Since $\langle I|C_1Q|\Phi_0\rangle$ in a level expansion is a sum of many terms which start with contribution of level 0 equal to $t = 0.5463\ldots$, we regard the values in the table as a reasonably good confirmation of our claims.

One can be puzzled about the last entry 0.0671 which seems to grow again. This value however depends on the values of level 10 components in the solution Φ_0 which are likely to be affected significantly by the fact that Φ_0 itself is found from the action truncated to level 10 fields at most.

7. Concluding remarks

One of the least explored aspects of the string field algebra is the issue of associativity. It has been known for a long time that some star products involving operators integrated over half of the string violate associativity [21, 43]. In this paper we have seen two new occasions where the problems might arise. First is when one keeps multiplying $L_{-2}|0\rangle$ by the vacuum $|0\rangle$ from the right. In few steps the level truncation will stop converging and associativity of the star product is recovered only after analytic continuation. Second case, where we definitely break the associativity is in the case of unbalanced wedge states, whose matter and ghost parts do not match. We have explicitly shown that in this case the three vertex loses its cyclicity, which implies the loss of associativity as well. Alternatively, one could imagine that some anomalous conservation laws receive some additional corrections when applied to wedge states, but this possibility is again in conflict with associativity. This problem is likely not just an academic question, since these anomalous conservation laws give nontrivial information about the classical solutions of the string field theory [44, 45].

[10] Thanks also to P.J. de Smet for some technical help in manipulating the solution while working together on another project.

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Another type of anomaly is the famous $c_0|I\rangle$ problem [18]. We have given a partial solution, but it would be nice to understand it more deeply and in a broader context.

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A. Star products in level expansion

In this appendix we would like to collect some numerical results showing, how well the level expansion works for star products. We have performed some explicit checks at level 20, where one of the states was particularly simple, and some other checks at level 16 which confirmed the composition law (2.29) obtained by the gluing ideas described in section 2. We have written for that purpose a computer program in *Mathematica* which is based on the Virasoro conservation laws for the three vertex [20, 3, 18].

A.1 Some level 20 calculations

First let us present the results, how good is the identity state $|I\rangle = |1\rangle$ acting on some basic states

$$\begin{aligned}
|0\rangle * |I\rangle &= |0\rangle + 0.00008 L_{-2}|0\rangle - 0.00007 L_{-3}|0\rangle - 0.00068 L_{-4}|0\rangle + \\
&\quad + 0.00039 L_{-2}L_{-2}|0\rangle + \cdots \\
L_{-2}|0\rangle * |I\rangle &= 0.9987 L_{-2}|0\rangle - 0.0001 L_{-3}|0\rangle - 0.0001 L_{-4}|0\rangle + \\
&\quad + 0.0007 L_{-2}L_{-2}|0\rangle + \cdots \\
L_{-2}L_{-2}|0\rangle * |I\rangle &= 0.0054 L_{-2}|0\rangle - 0.0001 L_{-3}|0\rangle + 0.0002 L_{-4}|0\rangle + \\
&\quad + 0.9967 L_{-2}L_{-2}|0\rangle + \cdots \\
L_{-4}|0\rangle * |I\rangle &= 0.0035 L_{-2}|0\rangle - 0.0005 L_{-3}|0\rangle + 0.9967 L_{-4}|0\rangle + \\
&\quad + 0.0002 L_{-2}L_{-2}|0\rangle + \cdots . \qquad (A.1)
\end{aligned}$$

Now let us present the results for the products of the vacuum $|0\rangle$ and other wedge states to verify the composition rule $|r\rangle * |s\rangle = |r+s-1\rangle$.

$$\begin{aligned}
|2\rangle * |3\rangle &= |0\rangle - 0.25006 L_{-2}|0\rangle + 0.00197 L_{-3}|0\rangle + 0.03132 L_{-4}|0\rangle + \\
&\quad + 0.03126 L_{-2}L_{-2}|0\rangle + \cdots \\
|2\rangle * |\infty\rangle &= |0\rangle - 0.32085 L_{-2}|0\rangle + 0.00563 L_{-3}|0\rangle + 0.03294 L_{-4}|0\rangle + \\
&\quad + 0.05137 L_{-2}L_{-2}|0\rangle + \cdots \\
|2\rangle * |1/2\rangle &= |0\rangle - 38723.7 L_{-2}|0\rangle - 22117.4 L_{-3}|0\rangle - 12233.8 L_{-4}|0\rangle + \\
&\quad + 34414.4 L_{-2}L_{-2}|0\rangle + \cdots . \qquad (A.2)
\end{aligned}$$

The first two products should be compared with the wedge states

$$\begin{aligned}|4\rangle &= |0\rangle - 0.25L_{-2}|0\rangle + 0.03125L_{-4}|0\rangle + 0.03125L_{-2}L_{-2}|0\rangle + \cdots \\ |\infty\rangle &= |0\rangle - 0.33333L_{-2}|0\rangle + 0.03333L_{-4}|0\rangle + 0.05556L_{-2}L_{-2}|0\rangle + \cdots .\end{aligned} \tag{A.3}$$

We see that the agreement is quite good (within 0.23%) for the state $|90^\circ\rangle$ but is considerably worse (within 7.5%) for the $|\infty\rangle$. The last product indicates that the states $|r\rangle$ with $r < 1$ do not have much sense in the level expansion.

A.2 Level 16 calculations

$$\begin{aligned}|I\rangle * |I\rangle &= |0\rangle + 1.00386L_{-2}|0\rangle - 0.50098L_{-4}|0\rangle + 0.49723L_{-2}L_{-2}|0\rangle + \cdots \\ |\infty\rangle * |\infty\rangle &= |0\rangle - 0.36150L_{-2}|0\rangle + 0.03338L_{-4}|0\rangle + 0.06549L_{-2}L_{-2}|0\rangle + \cdots \\ |\infty\rangle * |I\rangle &= |0\rangle - 0.32656L_{-2}|0\rangle + 0.00267L_{-3}|0\rangle + 0.03148L_{-4}|0\rangle + \\ &\quad + 0.05365L_{-2}L_{-2}|0\rangle + \cdots \\ |\infty\rangle * |3\rangle &= |0\rangle - 0.33434L_{-2}|0\rangle - 0.00564L_{-3}|0\rangle + 0.03394L_{-4}|0\rangle + \\ &\quad + 0.05587L_{-2}L_{-2}|0\rangle + \cdots \\ |1/2\rangle * |3\rangle &= |0\rangle - 2147.14L_{-2}|0\rangle + 1327.72L_{-3}|0\rangle - 553.046L_{-4}|0\rangle + \\ &\quad + 2074.33L_{-2}L_{-2}|0\rangle + \cdots \\ |3\rangle * |3\rangle &= |0\rangle - 0.28708L_{-2}|0\rangle + 0.03348L_{-4}|0\rangle + 0.04122L_{-2}L_{-2}|0\rangle + \cdots \\ |2\rangle * |3\rangle &= |0\rangle - 0.25008L_{-2}|0\rangle + 0.00246L_{-3}|0\rangle + 0.03135L_{-4}|0\rangle + \\ &\quad + 0.03127L_{-2}L_{-2}|0\rangle + \cdots \\ |2\rangle * |I\rangle &= |0\rangle + 0.00010L_{-2}|0\rangle - 0.00008L_{-3}|0\rangle - 0.00109L_{-4}|0\rangle + \\ &\quad + 0.00066L_{-2}L_{-2}|0\rangle + \cdots \\ |2\rangle * |\infty\rangle &= |0\rangle - 0.31966L_{-2}|0\rangle + 0.00668L_{-3}|0\rangle + 0.03293L_{-4}|0\rangle + \\ &\quad + 0.05096L_{-2}L_{-2}|0\rangle + \cdots \\ |2\rangle * |1/2\rangle &= |0\rangle - 1876.75L_{-2}|0\rangle - 1163.92L_{-3}|0\rangle - 567.608L_{-4}|0\rangle + \\ &\quad + 1851.82L_{-2}L_{-2}|0\rangle + \cdots .\end{aligned} \tag{A.4}$$

Let us compare these results obtained in level expansion with the exact answer. The errors for products with sliver state $|\infty\rangle$ are smallest for $|3\rangle$ state: 0.3% at level 2 and 1.8% at level 4 coefficients. The biggest error is when we multiply the sliver with another sliver $|\infty\rangle$. It is 8.4% at level 2 and 18% at level 4.

The errors for the product of $|I\rangle$ are again biggest for the $|\infty\rangle$ state with 2.1% at level 2 and 5.6% at level 4. The errors in the product of the identity with itself are 0.39% or 0.55% respectively.

The errors for $|0\rangle * |3\rangle$ are the lowest of all of the examples: 0.03% and 0.33% respectively.

The moral is that the wedge composition rule works better for states closer to the vacuum. It works worse for the identity and the worst for the $|\infty\rangle$ state.

B. Behavior of the wedge state coefficients

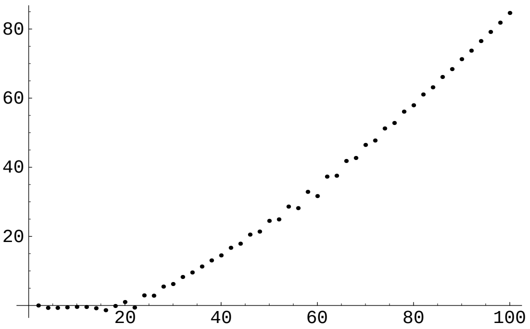

Figure 3: Plot of $\log |v_n|$, where v_n are coefficients appearing in the definition of the star algebra identity $|I\rangle = |1\rangle$. The odd values of n are omitted as all $v_{2k+1} = 0$ trivially. Despite the apparent exponential growth of the coefficients starting at level 20, the identity is well behaved in level expansion.

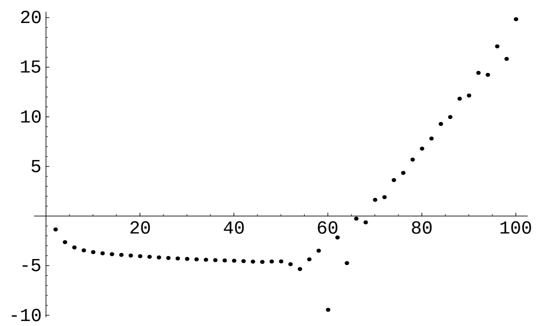

Figure 4: Plot of $\log |v_n|$ for the first 50 even coefficients in the definition of the wedge state $|3/2\rangle$. All the coefficients were calculated exactly as rational numbers, therefore the irregularity around $n = 60$ should be attributed to a chaotic behavior rather than to numerical errors.

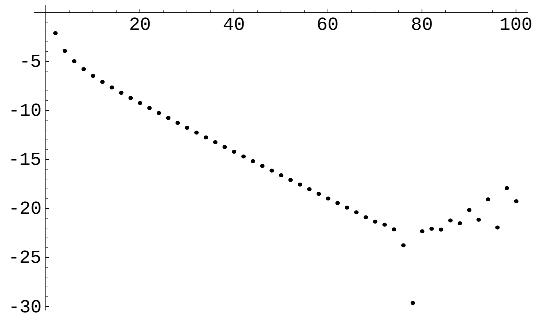

Figure 5: Plot of $\log |v_n|$ for the first 50 even coefficients in the definition of the wedge state $|5/2\rangle$. This state is quite close to the vacuum, this is the reason why the coefficients decrease exponentially to a rather high level $n \sim 80$.

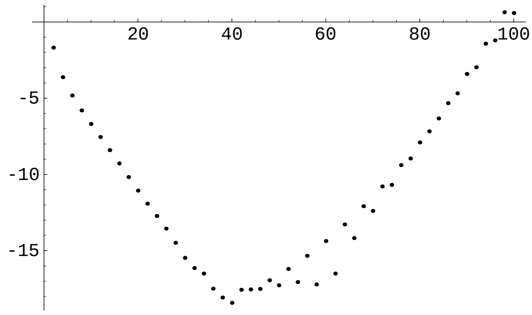

Figure 6: Plot of $\log|v_n|$ for the first 50 even coefficients in the definition of the wedge state $|3\rangle$.

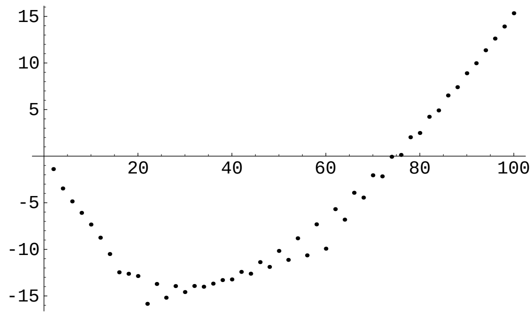

Figure 7: Plot of $\log|v_n|$ for the first 50 even coefficients in the definition of the wedge state $|4\rangle$.

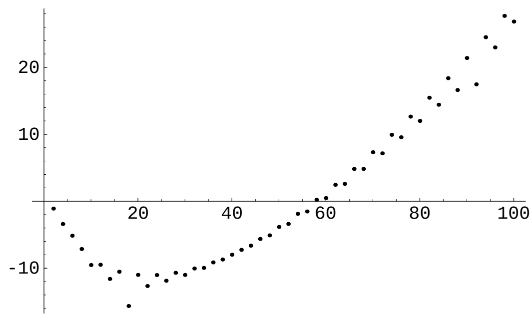

Figure 8: Plot of $\log|v_n|$ for the first 50 even coefficients in the definition of the sliver state $|\infty\rangle$.

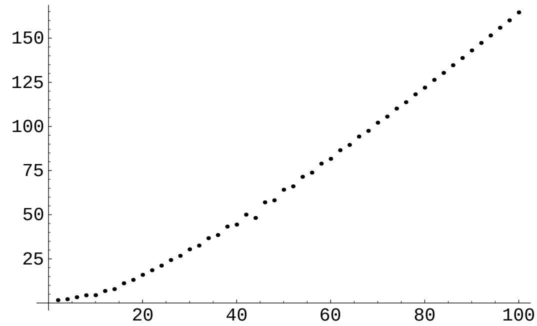

Figure 9: Plot of $\log |v_n|$ for the first 50 even coefficients in the definition of the wedge state $|\frac{1}{2}\rangle$. This state is meaningless from the geometric point of view of conformal field theory. Its star product with other states in level expansion does not converge as could be expected from the exponential growth of the coefficients.

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